Sensitivity Derivatives for Static Test Loading Boundary Conditions

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THE adjoint variable technique (known also as the dummy load technique[1]) is widely used for calculating sensitivity derivatives of stress and displacement constraints in structural optimization applications. It is more efficient than the direct method for obtaining these derivatives when the number of design variables is larger than the number of active displacement or stress constraints.

The application of the method is straightforward for simple displacement or stress boundary conditions. For more generalized boundary conditions it requires specialized treatment. Hsieh and Arora,[2] for example, employ Lagrange multipliers to extend the method to general displacement boundary conditions. The present Note is concerned with the application of the adjoint variable method under the special combination of displacement and stress boundary conditions which occurs when static test conditions are simulated.

In a typical loading apparatus the load is transmitted to the specimen through components which are much more rigid than the specimen itself. When the test is to be simulated by a finite element analysis it is possible to model both the specimen and the loading device and employ only force (or stress) boundary conditions. However, aside from the increased computational cost due to a more complex model, the problem becomes ill conditioned because of the disparity between the rigidities of the specimen and loading apparatus.

It, therefore, is preferable to solve the problem employing a two-stage process. First, unit displacements (in the direction of the load) are applied to a finite element model of the specimen. The internal stresses and displacements are calculated, as well as the total reaction force at the boundary. Next, the ratio between the applied load and the total reaction force due to unit displacements is calculated and the fields of internal stresses and displacements are multiplied by that ratio.

The partitioned form of the equilibrium equation for a structure discretized by a finite element model can be written as

$$\begin{bmatrix} K_{aa} & K_{ad} \\ K_{ad}^T & K_{dd} \end{bmatrix} \begin{Bmatrix} U_I \\ I \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_I \end{Bmatrix} \tag{1}$$

where K_{aa}, K_{ad}, and K_{dd} are submatrices of the stiffness matrix, U_I the unknown displacement vector per unit applied displacements, I the vector of unit boundary displacements, R_I the reaction vector per unit applied displacement, and $\boldsymbol{0}$ a null vector.

The total load applied to the specimen when the displacement vector I is specified is

$$f_I = I^T R_I \tag{2}$$

If the required applied load is f, then the total displacement vector U is

$$U = sU_I \tag{3}$$

where

$$s = f/f_I \tag{4}$$

Assume that the derivative of some function g of the displacement field (e.g., a stress component) with respect to a design parameter x is required. The function can also depend explicitly on the design parameter so that $g = g(x, U)$. The derivative of g with respect to x may be written as

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\partial g}{\partial x} + Z^T \frac{\mathrm{d}U}{\mathrm{d}x} \tag{5}$$

where Z denotes a vector with $z_i = \partial g/\partial u_i$. Using Eq. (3) we obtain

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\partial g}{\partial x} + Z^T \left[\frac{\mathrm{d}s}{\mathrm{d}x} U_I + s\frac{\mathrm{d}U_I}{\mathrm{d}x}\right] \tag{6}$$

To obtain $\mathrm{d}U_I/\mathrm{d}x$, differentiate Eq. (1) with respect to x

$$K_{aa}\frac{\mathrm{d}U_I}{\mathrm{d}x} = -\frac{\mathrm{d}K_{ad}}{\mathrm{d}x} I - \frac{\mathrm{d}K_{aa}}{\mathrm{d}x} U_I \tag{7}$$

$$\frac{\mathrm{d}R_I}{\mathrm{d}x} = \frac{\mathrm{d}K_{ad}^T}{\mathrm{d}x} U_I + K_{ad}^T \frac{\mathrm{d}U_I}{\mathrm{d}x} + \frac{\mathrm{d}K_{dd}}{\mathrm{d}x} I \tag{8}$$

Finally, to obtain $\mathrm{d}s/\mathrm{d}x$, differentiate Eqs. (2) and (4)

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\Big/ f_I - \frac{s}{f_I} I^T \frac{\mathrm{d}R_I}{\mathrm{d}x} \tag{9}$$

The direct method for obtaining $\mathrm{d}g/\mathrm{d}x$ starts by solving Eq. (7) for $\mathrm{d}U_I/\mathrm{d}x$, then obtaining $\mathrm{d}R_I/\mathrm{d}x$ from Eq. (8) and $\mathrm{d}s/\mathrm{d}x$ from Eq. (9), and, finally, using Eq. (6). The direct method has a disadvantage that the entire process of calculation has to be repeated for any other design variable.

To obtain the adjoint method we first transform Eq. (9) with the aid of Eqs. (1), (7), and (8)

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\Big/ f_I - \frac{s}{f_I}\left[2I^T \frac{\mathrm{d}K_{ad}^T}{\mathrm{d}x} U_I + I^T \frac{\mathrm{d}K_{dd}}{\mathrm{d}x} I + U_I^T \frac{\mathrm{d}K_{aa}}{\mathrm{d}x} U_I\right] \tag{10}$$

An adjoint variable Λ is now defined as the solution to

$$K_{aa}\Lambda = Z \tag{11}$$

Using Eqs. (7) and (11), Eq. (6) becomes

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\partial g}{\partial x} + \frac{\mathrm{d}s}{\mathrm{d}x} Z^T U_I - s\Lambda^T \left(\frac{\mathrm{d}K_{ad}}{\mathrm{d}x} I + \frac{\mathrm{d}K_{aa}}{\mathrm{d}x} U_I\right) \tag{12}$$

Now, $\mathrm{d}g/\mathrm{d}x$ can be evaluated from Eq. (12), where $\mathrm{d}s/\mathrm{d}x$ is given by Eq. (10). The major difference between the adjoint variable and the direct method is that instead of solving Eq.

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(7) for dU_l/dx one solves Eq. (11) for Λ. For large systems the solution process for these equations is the major computational cost in the derivative calculation. Because Eq. (7) depends only on the design variable x and Eq. (11) on the constraint g, the use of the adjoint method is preferred when the number of design variables is larger than the number of constraints. The direct method is better when the number of constraints is larger than the number of design variables.

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References

[1] Haug, E. J. and Arora, J. S., *Applied Optimal Design*, John Wiley and Sons, New York, 1979.

[2] Hsieh, C. C. and Arora, J. S., "Structural Design Sensitivity Analysis with General Boundary Conditions," University of Iowa Tech. Rept. CAD-SS-83.5, March 1983.